

PARTIAL DEHN TWISTS OF FREE GROUPS RELATIVE TO LOCAL DEHN TWISTS - A DICHOTOMY

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ABSTRACT. A criterion for quadratic or higher growth of group automorphisms is established which are represented by graph-of-groups automorphisms with certain well specified properties.

As a consequence, it is derived (using results of a previous paper of the author) that every partial Dehn twist automorphism of F_n relative to local Dehn twist automorphisms is either an honest Dehn twist automorphism, or else has quadratic growth.

1. INTRODUCTION

Dehn twist are well known from surface homeomorphisms: Any set \mathcal{C} of pairwise disjoint essential closed curves c_i on a surface S , together with a set of *twist exponents* $n_i \in \mathbb{Z}$, defines a homeomorphism $h : S \rightarrow S$ through “twisting” S along each c_i precisely n_i times.

The set \mathcal{C} defines canonically a *dual* graph-of-groups \mathcal{G} with isomorphism $\pi_1 \mathcal{G} \cong \pi_1 S$, where the vertex groups of \mathcal{G} are the fundamental groups of the components of $S \setminus \mathcal{C}$ and the edge groups are isomorphic to \mathbb{Z} . The automorphism $h_* : \pi_1 S \rightarrow \pi_1 S$ induced by the multi-Dehn-twist h can be described algebraically through a graph-of-groups isomorphism $H : \mathcal{G} \rightarrow \mathcal{G}$.

This natural correspondence between geometric and algebraic data has given rise to a more general definition of a *Dehn twist automorphism* φ of a group G , via a graph-of-groups \mathcal{G} equipped with an isomorphism $\pi_1 \mathcal{G} \cong G$ and a graph-of-groups isomorphism $H : \mathcal{G} \rightarrow \mathcal{G}$, which satisfy extra conditions that mimic the above described surface situation, so that one gets $H_* = \varphi$ (up to inner automorphisms).

A special case, which is useful in many circumstances, is given by requiring that H acts trivially on the underlying graph $\Gamma(\mathcal{G})$ and on each of the vertex groups G_v of \mathcal{G} , and that furthermore all edge groups of \mathcal{G} are trivial. In this case the automorphism $H_* : \pi_1 \mathcal{G} \rightarrow \pi_1 \mathcal{G}$ is determined by the family of *correction terms* $\delta_e \in G_{\tau(e)}$ for any edge e of \mathcal{G} , where $\tau(e)$ denotes the terminal vertex of e .

In the paper [10] the more general case of a *partial Dehn twist* $H : \mathcal{G} \rightarrow \mathcal{G}$ relative to a subset \mathcal{V} of vertices of \mathcal{G} has been investigated, which differs

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from the above described situation in that for any $v \in \mathcal{V}$ the induced vertex group automorphism $H_v : G_v \rightarrow G_v$ may not be the identity.

If for each $v \in \mathcal{V}$ the map $H_v : \mathcal{G}_v \rightarrow \mathcal{G}_v$ is itself a Dehn twist automorphism, then $H : \mathcal{G} \rightarrow \mathcal{G}$ is called a *partial Dehn twist relative to a family of local Dehn twists*. It is shown in [10] that in this case H can be *blown-up* to a refined graph-of-groups isomorphism which is a Dehn twist that incorporates both, H and the family of all H_v , provided that the following criterion is satisfied:

Criterion: For every edge e of H with endpoint $v \in \mathcal{V}$ the correction term δ_e is H_v -zero.

Here for any graph-of-groups automorphism $H : \mathcal{G} \rightarrow \mathcal{G}$, the associated path group $\Pi(\mathcal{G})$, and any vertex v of \mathcal{G} an element $g \in \pi_1(\mathcal{G}, v) \subset \Pi(\mathcal{G})$ is H -zero if and only if there exists an element $h \in \Pi(\mathcal{G})$ such that $h^{-1}gH_{*v}(h)$ has \mathcal{G} -length 0.

The main result of this paper is to show that this sufficient criterion is also necessary. In fact, we show:

Theorem 1.1. *Let $H : \mathcal{G} \rightarrow \mathcal{G}$ be a partial Dehn twist relative to a subset \mathcal{V} of vertices of \mathcal{G} . Assume for some $v \in \mathcal{V}$ that the vertex group G_v is free, and that $H_v : G_v \rightarrow G_v$ is a Dehn twist automorphisms.*

If there is an edge e of \mathcal{G} with correction term $\delta_e \in G_v$ that is not H_v -zero, then the automorphism $H_ : \pi_1\mathcal{G} \rightarrow \pi_1\mathcal{G}$ has at least quadratic growth.*

Since Dehn twist automorphisms are known to have linear growth, this shows that H_* is not conjugate to any Dehn twist automorphism. In particular, H can indeed not be blown up via the local Dehn twists H_v to obtain a global Dehn twist of $\pi_1\mathcal{G}$.

By combining this theorem with the main result of [10], we obtain:

Corollary 1.2. *Let $\hat{\varphi} \in \text{Out}(F_n)$ be represented by a partial Dehn twist relative to a family of local Dehn twists.*

Then either $\hat{\varphi}$ is itself a Dehn twist automorphism, or else $\hat{\varphi}$ has at least quadratic growth.

The proof of this corollary is algorithmic, i.e. it can be effectively decided which alternative of the stated dichotomy holds. This is a crucial ingredient in the author's work [11], where an algorithm is given that decides whether a polynomial growth automorphism of a free group F_n is, up to passing to a power, induced by a surface homeomorphism. It is also the starting point of a more detailed analysis of the growth of conjugacy classes for polynomially growing automorphisms of F_n , see [11].

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2. GRAPHS-OF-GROUPS AND THEIR ISOMORPHISMS

The purpose of this and the following section is to briefly recall some basic knowledge and to establish some preliminary lemmas about graph-of-groups, Dehn twists on graph-of-groups, efficient Dehn twist as well as the notion of H -conjugation, which is introduced in [10].

2.1. Basics on graphs-of-groups.

Most of our notations are taken from [3]; we refer the readers to [8], [7] and [1] for more detailed informations and discussions.

Throughout this paper, we refer to a *graph* as a finite, non-empty, connected graph in the sense of Serre (cf. [8]).

For a graph Γ , we denote by $V(\Gamma)$, $E(\Gamma)$ its *vertex set* and *edge set* respectively. For an edge $e \in E(\Gamma)$, we denote by $\tau(e)$ its *terminal vertex* and $\tau(\bar{e})$ its *initial vertex*. The *inverse* of an edge e is denoted by \bar{e} .

Notice in particular that our graph Γ is non-oriented while one can always choose an *orientation* $E^+(\Gamma) \subset E(\Gamma)$, which satisfies that $E^+(\Gamma) \cup \bar{E}^+(\Gamma) = E(\Gamma)$ and $E^+(\Gamma) \cap \bar{E}^+(\Gamma) = \emptyset$, where $\bar{E}^+(\Gamma) = \{\bar{e} \mid e \in E^+(\Gamma)\}$.

Definition 2.1. A *graph-of-groups* \mathcal{G} is defined by

$$\mathcal{G} = (\Gamma, (G_v)_{v \in V(\Gamma)}, (G_e)_{e \in E(\Gamma)}, (f_e)_{e \in E(\Gamma)})$$

where:

- (1) Γ is a graph, called the *underlying graph*;
- (2) each G_v is a group, called the *vertex group* of v ;
- (3) each G_e is a group, called the *edge group* of e , and we require $G_e = G_{\bar{e}}$ for every $e \in E(\Gamma)$;
- (4) for each $e \in E(\Gamma)$, the map $f_e : G_e \rightarrow G_{\tau(e)}$ is an injective *edge homomorphism*.

Unless otherwise stated, in this paper we will always assume that all vertex and all edge groups of any graph-of-groups \mathcal{G} are finitely generated.

Given a graph-of-groups \mathcal{G} , we usually denote by $\Gamma(\mathcal{G})$ the graph underlying it. The vertex set of $\Gamma(\mathcal{G})$ is denoted by $V(\mathcal{G})$ while the edge set is denoted by $E(\mathcal{G})$.

Definition 2.2. The *word group* $W(\mathcal{G})$ of a graph-of-groups \mathcal{G} is the free product of vertex groups and the free group generated by *stable letters* $(t_e)_{e \in E(\Gamma)}$, i.e. $W(\mathcal{G}) = *(G_v)_{v \in V(\Gamma)} * F(\{t_e; e \in E(\Gamma)\})$.

The *path group* (sometimes also called *Bass group*) of \mathcal{G} is defined by $\Pi(\mathcal{G}) = W(\mathcal{G})/R$, where R is the normal subgroup subjects to the following relations:

- $\diamond t_e = t_{\bar{e}}^{-1}$, for every $e \in E(\Gamma)$;
- $\diamond f_{\bar{e}}(g) = t_e f_e(g) t_{\bar{e}}^{-1}$, for every $e \in E(\Gamma)$ and every $g \in G_e$.

Remark 2.3. A word $w \in W(\mathcal{G})$ can always be written in the form $w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q$ ($q \geq 0$), where each $t_i \in F(\{t_e; e \in E(\Gamma)\})$ stands for the stable letter of the edge e_i and each $r_i \in *(G_v)_{v \in V(\Gamma)}$.

The sequence (t_1, t_2, \dots, t_q) is called the *path type* of w , the number q is called the *path length* of w . In this case, we say that $e_1 e_2 \dots e_q$ is the path underlying w . Two path types (t_1, t_2, \dots, t_q) and $(t'_1, t'_2, \dots, t'_s)$ are said to be same if and only if $q = s$ and $t_i = t'_i$ for each $1 \leq i \leq q$.

Definition 2.4. Let $w \in W(\mathcal{G})$ be a word of the form $w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q$. The word w is said to be *connected* if $r_0 \in G_{\tau(\bar{e}_1)}$, $r_q \in G_{\tau(e_q)}$, and $\tau(e_i) = \tau(\bar{e}_{i+1})$, $r_i \in G_{\tau(e_i)}$, for $i = 1, 2, \dots, q-1$.

Moreover, if w is connected and $\tau(e_q) = \tau(\bar{e}_1)$, we say that w is a *closed connected word issued at the vertex* $\tau(e_q)$.

Definition 2.5. Let $w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q \in W(\mathcal{G})$, w is said to be *reduced* if it satisfies:

- ◊ if $q = 0$, then $w = r_0$ isn't equal to the unit element;
- ◊ if $q > 0$, then whenever $t_i = t_{i+1}^{-1}$ for some $1 \leq i \leq q-1$ we have $r_i \notin f_{e_i}(G_{e_i})$.

Moreover the word w is said to be *cyclically reduced* if it is reduced and if $q > 0$ and $t_1 = t_q^{-1}$, then $r_q r_0 \notin f_{e_q}(G_{e_q})$.

We recall the following facts.

Proposition 2.6. For any graph-of-groups \mathcal{G} , the following holds:

- (1) Every non-trivial element of $\Pi(\mathcal{G})$ can be represented as a reduced word.
- (2) Every reduced word is a non-trivial element in $\Pi(\mathcal{G})$.
- (3) If $w_1, w_2 \in W(\mathcal{G})$ are two reduced words representing the same element in $\Pi(\mathcal{G})$, then w_1 and w_2 are of the same path type. In particular, w_2 is connected if and only if w_1 is connected.

Definition 2.7. (fundamental groups)

1. Fundamental groups based at v_0

For any $v_0 \in V(\Gamma)$, the *fundamental group based at v_0* , denoted by $\pi_1(\mathcal{G}, v_0)$, consists of the elements in $\Pi(\mathcal{G})$ that are closed connected words issued at v_0 .

For a vertex $w_0 \in V(\Gamma)$ different from v_0 , we have $\pi_1(\mathcal{G}, v_0) \cong \pi_1(\mathcal{G}, w_0)$. In fact, let $W \in \Pi(\mathcal{G})$ be a connected word with underlying path from v_0 to w_0 . The restriction of $ad_W : \Pi(\mathcal{G}) \rightarrow \Pi(\mathcal{G})$ to $\pi_1(\mathcal{G}, w_0)$ induces an isomorphism from $\pi_1(\mathcal{G}, w_0)$ to $\pi_1(\mathcal{G}, v_0)$. Sometimes we write $\pi_1(\mathcal{G})$ when the choice of basepoint doesn't make a difference.

2. Fundamental groups at a maximal tree T_0

The *fundamental group at T_0* , denoted by $\pi_1(\mathcal{G}, T_0)$, is generated by the groups G_v , for all $v \in V(\Gamma)$, and the elements t_e , for all $e \in E(\Gamma)$, subjects to the relations:

- ◊ $t_e^{-1} = t_{\bar{e}}, t_e f_e(g) t_e^{-1} = f_{\bar{e}}(g)$, for $e \in E(\Gamma)$, $g \in G_e$;
- ◊ $t_e = 1$, for $e \in E(T_0)$.

By definition we have immediately

$$\pi_1(\mathcal{G}, T_0) = \Pi(\mathcal{G}) / \ll t_e = 1, e \in E(T_0) \gg.$$

It's shown in the book of Serre [8] that the above two definitions of fundamental groups are equivalent.

Theorem 2.8 (Serre). *For a graph-of-groups \mathcal{G} , let v_0 be a vertex and T_0 a maximal tree. Then $\pi_1(\mathcal{G}, v_0) \cong \pi_1(\mathcal{G}, T_0)$.*

It follows immediately that, for a graph-of-groups \mathcal{G} with trivial edge groups, the product $\ast(G_v)_{v \in V(\mathcal{G})}$ is free and forms a free factor of $\pi_1(\mathcal{G})$, moreover the disjoint union of basis of each vertex group $\bigcup_{v \in V(\mathcal{G})} \mathcal{B}_v$ is a subset of the basis of $\pi_1(\mathcal{G})$.

Definition 2.9. [Graph-of-groups Isomorphisms]

Let $\mathcal{G}_1, \mathcal{G}_2$ be two graphs of groups. Denote $\Gamma_1 = \Gamma(\mathcal{G}_1)$ and $\Gamma_2 = \Gamma(\mathcal{G}_2)$. An isomorphism $H : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a tuple of the form

$$H = (H_\Gamma, (H_v)_{v \in V(\Gamma_1)}, (H_e)_{e \in E(\Gamma_1)}, (\delta(e))_{e \in E(\Gamma_1)})$$

where

- (1) $H_\Gamma : \Gamma_1 \rightarrow \Gamma_2$ is a graph isomorphism;
- (2) $H_v : G_v \rightarrow G_{H_\Gamma(v)}$ is a group isomorphism, for any $v \in V(\Gamma_1)$;
- (3) $H_e = H_{\bar{e}} : G_e \rightarrow G_{H_\Gamma(e)}$ is a group isomorphism, for any $e \in E(\Gamma_1)$;
- (4) for every $e \in E(\Gamma_1)$, the *correction term* $\delta(e) \in G_{\tau(H_\Gamma(e))}$ is an element such that

$$H_{\tau(e)} f_e = \text{ad}_{\delta(e)} f_{H_\Gamma(e)} H_e.$$

Remark 2.10. A graph-of-groups isomorphism $H : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ induces an isomorphism $H_* : \Pi(\mathcal{G}_1) \rightarrow \Pi(\mathcal{G}_2)$ defined on the generators by:

$$\begin{aligned} H_*(g) &= H_v(g), \text{ for } g \in G_v, v \in V(\Gamma_1); \\ H_*(t_e) &= \delta(\bar{e}) t_{H_\Gamma(e)} \delta(e)^{-1}, \text{ for } e \in E(\Gamma_1). \end{aligned}$$

It's easy to verify by computation that H_* preserves the relations $t_e t_{\bar{e}} = 1$ for any $e \in E(\mathcal{G})$ and $f_{\bar{e}}(g) = t_e f_e(g) t_e^{-1}$, for any $e \in E(\mathcal{G})$ and $g \in G_e$.

Furthermore, the restriction of H_* to $\pi_1(\mathcal{G}_1, v)$, where $v \in V(\Gamma_1)$, is also an isomorphism, denoted by $H_{*v} : \pi_1(\mathcal{G}_1, v) \rightarrow \pi_1(\mathcal{G}_2, H_\Gamma(v))$.

Remark 2.11. As in [3], we define the *outer isomorphism* induced by a group isomorphism $f : G_1 \rightarrow G_2$ as the equivalence class

$$\hat{f} = \{\text{ad}_g f : G_1 \rightarrow G_2 \mid g \in G_2\}.$$

Hence H_{*v} induces an outer isomorphism $\hat{H}_{*v} : \pi_1(\mathcal{G}_1, v) \rightarrow \pi_1(\mathcal{G}, H_\Gamma(v))$.

Observe that when choosing a different vertex v_1 as basepoint, we may choose a word $W \in \Pi(\mathcal{G}_1)$ with underlying path from v_1 to v to obtain the

following commutative diagram:

$$\begin{array}{ccc} \pi_1(\mathcal{G}_1, v) & \xrightarrow{H_{*v}} & \pi_1(\mathcal{G}_2, H_\Gamma(v)) \\ ad_W \downarrow & & ad_{H_*(w)} \downarrow \\ \pi_1(\mathcal{G}_1, v_1) & \xrightarrow{H_{*v_1}} & \pi_1(\mathcal{G}_2, H_\Gamma(v_1)) \end{array}$$

By Lemma 2.2 and Lemma 3.10 in [3], \hat{H}_{*v} determines an outer isomorphism $\hat{H}_{*v_1} : \pi_1(\mathcal{G}_1, v_1) \rightarrow \pi_1(\mathcal{G}_2, H_\Gamma(v_1))$.

In this sense, we observe that $H : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ induces an outer isomorphism $\hat{H} : \pi_1(\mathcal{G}_1) \rightarrow \pi_1(\mathcal{G}_2)$ which doesn't depend on the choice of basepoint.

2.2. H-conjugation.

We recall in this subsection some basic definitions and properties about the notion of H -conjugation. Contrary to the previous subsection, which only contained standard definitions and notation, the content of this subsection has been defined in [10] and to our knowledge didn't exist previously.

Definition 2.12. For a graph-of-groups automorphism $H : \mathcal{G} \rightarrow \mathcal{G}$ two reduced words $w_1, w_2 \in \Pi(\mathcal{G})$ are said to be H -conjugate to each other if there exists a reduced word $w \in \Pi(\mathcal{G})$ such that $w_1 = ww_2H_*(w)^{-1}$.

It's easy to show that H -conjugation is a well-defined equivalence relation on $\Pi(\mathcal{G})$.

Denote by $[w]_H$ the set which consists of all elements in $\Pi(\mathcal{G})$ that are H -conjugate to w . We call $[w]_H$ the H -conjugacy class of w .

Recall that the *path length* of a word $w \in \Pi(\mathcal{G})$ equals to the number of edges the path underlying w crosses. We denote the path length of w by $\|w\|_{\mathcal{G}}$.

Definition 2.13. A reduced word $w \in \Pi(\mathcal{G})$ is said to be H -minimal if it has the shortest path length among its H -conjugates. More specifically, if w is H -minimal, then for every $w_0 \in \Pi(\mathcal{G})$, $\|w_0wH_*(w_0)^{-1}\|_{\mathcal{G}} \geq \|w\|_{\mathcal{G}}$.

Since $\|w\|_{\mathcal{G}}$ is a natural number, one has that every reduced word $w \in \Pi(\mathcal{G})$ has a H -conjugate which is H -minimal.

Therefore there exists a well defined H -length:

$$\|w\|_{\mathcal{G},H} = \min\{\|gwH_*(g)^{-1}\|_{\mathcal{G}} \mid g \in \Pi(\mathcal{G})\}.$$

Moreover, $\|w\|_{\mathcal{G},H} = \|w\|_{\mathcal{G}}$ if and only if w is H -reduced.

Definition 2.14. A reduced word $w \in \Pi(\mathcal{G})$ is called H -zero if and only if its H -length equals to zero, i.e. $\|w\|_{\mathcal{G},H} = 0$.

It also can be shown that $w \in \Pi(\mathcal{G})$ is H -minimal if and only if it is H -reduced, as defined below:

Definition 2.15. Let $w \in \Pi(\mathcal{G})$ be a reduced word in the form of $w = r_0t_1r_1\dots r_{q-1}t_qr_q$, w is said to be H -reduced if its cannot be shortened by the elementary operation $w \mapsto w_1 = (r_0t_1)^{-1}wH_*(r_0t_1)$, i.e. $\|w\|_{\mathcal{G}} = \|w_1\|_{\mathcal{G}}$.

Remark 2.16. If $w' = r_0 t_1 r_1 \dots r_{k-1} t_k r_k \in [w]_H$ is H -reduced, then

$$\begin{aligned} w'_1 &= (r_0 t_1)^{-1} w' H_*(r_0 t_1) \\ &\dots \\ w'_i &= (r_{i-1} t_i)^{-1} w'_{i-1} H_*(r_{i-1} t_i) \\ &\dots \\ w'_k &= (r_{k-1} t_k)^{-1} w'_{k-1} H_*(r_{k-1} t_k) \end{aligned}$$

are also H -reduced.

Moreover, since H_* preserves the path lengths of reduced words, we also have $H_*(w') = w'^{-1} w' H_*(w')$, $H_*^{-1}(w') = H_*^{-1}(w') w' w'^{-1}$ are H -reduced.

Hence $\bigcup_{i=-\infty}^{+\infty} \{H_*^i(w'), H_*^i(w'_1), \dots, H_*^i(w'_k)\}$ covers all path types of H -reduced words in $[w]_H$.

Lemma 2.17. *Given a graph-of-groups isomorphism $H : \mathcal{G} \rightarrow \mathcal{G}$ which acts trivially on the underlying graph $\Gamma = \Gamma(\mathcal{G})$. Choose an arbitrary vertex $v_0 \in V(\Gamma)$ as basepoint. For every closed reduced word $W \in \pi_1(\mathcal{G}, v_0)$, there exist a vertex $v_1 \in V(\Gamma)$ and a reduced word $\gamma \in \Pi(\mathcal{G})$ which underlies a path from v_0 to v_1 such that $\gamma^{-1} W H(\gamma) \in \pi_1(\mathcal{G}, v_1)$ is H -reduced.*

Proof. In general, for every reduced word $W \in \Pi(\mathcal{G})$, there exists $\gamma \in \Pi(\mathcal{G})$ such that $\gamma^{-1} W H(\gamma)$ is H -reduced. In the case where H acts trivially on the graph Γ , the reduced words γ and $H(\gamma)$ underly exactly the same edge path. Hence the word $\gamma^{-1} W H(\gamma)$ is a closed word issued at the terminal vertex of γ .

Moreover, it derives from Section 2.2 that we can find such an H -reduced word with path type that is a subsequence of the path type of W by applying the elementary operation defined in Definition 2.15. \square

3. DEHN TWISTS

3.1. Classical Dehn twist.

Definition 3.1. [Classical Dehn twist] A *Dehn twist* D of a graph-of-groups \mathcal{G} is a graph-of-groups automorphism $D : \mathcal{G} \rightarrow \mathcal{G}$ which satisfies (where $\Gamma = \Gamma(\mathcal{G})$ denotes as usually the underlying graph):

- (1) $D_\Gamma = id_\Gamma$;
- (2) $D_v = id_{G_v}$, for all $v \in V(\Gamma)$;
- (3) $D_e = id_{G_e}$, for all $e \in E(\Gamma)$;
- (4) for each G_e , there is an element $\gamma_e \in Z(G_e)$ such that the correction term satisfies $\delta(e) = f_e(\gamma_e)$, where $Z(G_e)$ denotes the center of G_e .

We denote a Dehn twist defined as above by $D = D(\mathcal{G}, (\gamma_e)_{e \in E(\mathcal{G})})$

Remark 3.2 (Twistor). Given a Dehn twist $D = D(\mathcal{G}, (\gamma_e)_{e \in E(\mathcal{G})})$, we define the *twistor* of an edge $e \in E(\Gamma)$ by setting $z_e = \gamma_{\bar{e}}\gamma_e^{-1}$. Then for any edge e we have $z_e \in Z(G_e)$ and $z_{\bar{e}} = \gamma_e\gamma_{\bar{e}}^{-1} = z_e^{-1}$.

Remark 3.3. The induced automorphism $D_* : \Pi(\mathcal{G}) \rightarrow \Pi(\mathcal{G})$ is defined on generators as follows:

$$\begin{aligned} D_*(g) &= g, \text{ for } g \in G_v, v \in V(\Gamma); \\ D_*(t_e) &= t_e f_e(z_e), \text{ for every } e \in E(\Gamma). \end{aligned}$$

In particular, the induced automorphism on the fundamental group, $D_{*v} : \pi_1(\mathcal{G}, v) \rightarrow \pi_1(\mathcal{G}, v)$ where $v \in V(\Gamma)$, is called a *Dehn Twist automorphism*.

Definition 3.4. In general, a group automorphism $\varphi : G \rightarrow G$ is said to be a *Dehn twist automorphism* if it is represented by a graph-of-groups Dehn twist. More precisely, there exists a graph-of-groups \mathcal{G} , a vertex v of $\Gamma(\mathcal{G})$, a Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$, and an isomorphism $\theta : G \rightarrow \pi_1(\mathcal{G}, v)$ such that $\varphi = \theta^{-1} \circ D_{*v} \circ \theta$.

In this case the induced outer automorphism $\hat{\varphi} : G \rightarrow G$ is called a *Dehn twist outer automorphism*.

Remark 3.5. The reader may notice the following subtlety in the above definitions:

Because of the role of the base point v in Definition 3.4, it may well occur that two automorphisms φ_1 and φ_2 of a group G define the same outer automorphism $\hat{\varphi}_1 = \hat{\varphi}_2$ which is a Dehn twist outer automorphism, but only φ_1 is a Dehn twist automorphism, while φ_2 isn't.

Proposition 3.6 (Proposition 5.4 [3]). *Suppose \mathcal{G} is a graph-of-groups which satisfies that for every edge e there is an element $r_e \in G_{\tau(e)}$ with*

$$f_e(G_e) \cap r_e f_e(G_e) r_e^{-1} = \{1\}.$$

Then two Dehn twists $D = (\mathcal{G}, (\gamma_e)_{e \in E(\mathcal{G})})$, $D' = (\mathcal{G}, (\gamma'_e)_{e \in E(\mathcal{G})})$ determine the same outer automorphism of $\pi_1(\mathcal{G})$ if and only if $z_e = z'_e$ for all $e \in E(\Gamma)$.

This proposition shows that in many situations a Dehn twist on a given graph-of-groups is uniquely determined by its twistors. Thus sometimes we may define a Dehn twist by its twistors $(z_e)_{e \in E(\Gamma)}$ (for each $e \in E(\Gamma)$, $z_e \in Z(G_e)$ and $z_{\bar{e}} = z_e^{-1}$). In this case, we may conversely define:

$$\gamma_e = \begin{cases} z_e^{-1}, & e \in E^+(\Gamma) \\ 1, & e \in E^-(\Gamma). \end{cases}$$

3.2. General and partial Dehn twists.

As discussed in [10], we can define a Dehn twist in a slightly more general context by replacing the last condition of Definition 3.1 by the following:

- (4*) the correction term $\delta(e) \in C(f_e(G_e))$, where $C(f_e(G_e))$ denotes the centralizer of $f_e(G_e)$ in $G_{\tau(e)}$, for all $e \in E(\Gamma)$.

We call a graph-of-groups automorphism which satisfies conditions (1) – (3) in Definition 3.1 and (4*) a *general Dehn twist*.

It's shown in [10] that Dehn twists defined in either, the classical or the general version, are equivalent in the sense that: (i) every classical Dehn twist is a general Dehn twist; (ii) every general Dehn twist has a naturally corresponding classical Dehn twist which induces same outer automorphism.

On other hand, if \mathcal{G} is a graph-of-groups with trivial edge groups, and $H : \mathcal{G} \rightarrow \mathcal{G}$ is an automorphism which acts trivially on the graph and vertex groups, then it follows immediately from the above definitions that H induces Dehn twist automorphisms, for any choice of the family of correction terms.

Definition 3.7. (a) A *partial Dehn twist relative to a subset of vertices* $\mathcal{V} = \{v_1, v_2, \dots, v_m\}$ of \mathcal{G} is a graph-of-groups automorphism $H : \mathcal{G} \rightarrow \mathcal{G}$ such that

- (1) for every $e \in E(\mathcal{G})$ with $\tau(e) \in \mathcal{V}$, the edge group G_e is trivial;
- (2) H satisfies all conditions of a general Dehn twist except at $v_i \in \mathcal{V}$.

That is to say, any of vertex group automorphism H_{v_i} with $v_i \in \mathcal{V}$ may not be trivial.

(b) More specifically, a *partial Dehn twist relative to a family of local Dehn twists* is a partial Dehn twist relative to a subset of vertices \mathcal{V} of \mathcal{G} , and at any vertex $v \in \mathcal{V}$ the vertex group automorphism $H_v : G_v \rightarrow G_v$ is a Dehn twist automorphism.

Remark 3.8. If some automorphism $\varphi : G \rightarrow G$ is represented by a partial Dehn twist $H : \mathcal{G} \rightarrow \mathcal{G}$ relative to a family of vertices $v_i \in \mathcal{V} \subset V(\Gamma(\mathcal{G}))$, where every vertex group automorphism $H_{v_i} : G_{v_i} \rightarrow G_{v_i}$ induces an outer Dehn twist automorphism \hat{H}_{v_i} , then φ is a partial Dehn twist relative to a family of local Dehn twists.

This can be seen through replacing H by $J \circ H$, where the graph-of-groups automorphism $J : \mathcal{G} \rightarrow \mathcal{G}$ is the identity on all edge groups and on all vertex groups for vertices outside \mathcal{V} , and an inner automorphism on all G_{v_i} if $v_i \in \mathcal{V}$. Here the correction terms δ_e^J for edges e with terminal vertex $\tau(e) \notin \mathcal{V}$ are trivial, while for edges e with $\tau(e) \in \mathcal{V}$ they are properly chosen to “undo” the inner automorphism $J_{\tau(e)}$ on $G_{\tau(e)}$, so that for any $v \notin \mathcal{V}$ the induced automorphism $J_{*v} : \pi_1(\mathcal{G}, v) \rightarrow \pi_1(\mathcal{G}, v)$ is the identity map (and thus for any $v \in \mathcal{V}$ the automorphism J_{*v} is an inner automorphism). See Section 2.4 in [10] for more details.

3.3. Efficient Dehn Twist.

Unless otherwise stated, in this subsection we always assume $D : \mathcal{G} \rightarrow \mathcal{G}$ is a Dehn twist defined in the classical meaning. We write $D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$.

Two edges e_1 and e_2 with common terminal vertex v are called

- ◇ *positively bonded*, if there exist $n_1, n_2 \geq 1$ such that $f_{e_1}(z_{e_1}^{n_1})$ and $f_{e_2}(z_{e_2}^{n_2})$ are conjugate in G_v .

- ◇ *negative bonded*, if there exist $n_1 \geq 1$, $n_2 \leq 1$ such that $f_{e_1}(z_{e_1}^{n_1})$ and $f_{e_2}(z_{e_2}^{n_2})$ are conjugate in G_v .

For the rest of this subsection, we always assume for a graph-of-groups \mathcal{G} its fundamental group $\pi_1(\mathcal{G})$ is free and of finite rank $n \geq 2$. This implies, by definition of a classical Dehn twist, that any edge e with non-trivial twistor z_e has edge group $G_e \cong \mathbb{Z}$.

Definition 3.9 (Efficient Dehn twist [3]). A Dehn twist $D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ is said to be *efficient* if :

The graph-of-groups \mathcal{G} satisfies

- (1) \mathcal{G} is *minimal*: if $v = \tau(e)$ is a valence-one vertex, then the edge homomorphism $f_e : G_e \rightarrow G_v$ is not surjective.
- (2) There is no *invisible vertex*: there is no valence-two vertex $v = \tau(e_1) = \tau(e_2)$ ($e_1 \neq e_2$) such that both edge maps $f_{e_i} : G_{e_i} \rightarrow G_v$ ($i = 1, 2$) are surjective.
- (3) No *proper power*: if $r^p \in f_e(G_e)$ ($p \neq 0$) then $r \in f_e(G_e)$, for all $e \in E(\Gamma)$.

And together with the collection of twistors $(z_e)_{e \in E(\Gamma)}$, it also satisfies:

- (4) No *unused edge*: for every $e \in E(\Gamma)$, the twistor $z_e \neq 1$ (or equivalently $\gamma_e \neq \gamma_{\bar{e}}$).
- (5) If $v = \tau(e_1) = \tau(e_2)$, then e_1 and e_2 are not positively bonded.

The following has been shown in [3]:

Proposition 3.10. *For every vertex $v \in V(\mathcal{G})$ of an efficient Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$ the group G_v has rank*

$$\text{rk}(G_v) \geq 2.$$

□

Proposition 3.11. *Let $D : \mathcal{G} \rightarrow \mathcal{G}$ be an efficient Dehn twist, and let $v \in V(\mathcal{G})$ be any vertex.*

- (a) *A conjugacy class $[w]$ of $\pi_1(\mathcal{G}, v)$ is fixed by \widehat{D}_{*v} if and only if w has H -length*

$$\|w\|_{\mathcal{G}, H} = 0.$$

- (b) *An element $w \in \pi_1(\mathcal{G}, v)$ is fixed by D_{*v} if and only if $w \in G_v$.* □

It's shown in [3] that every Dehn twist (classical or general) can be transformed algorithmically into an efficient Dehn twist, and furthermore, the latter is essentially unique:

Theorem 3.12. (1) *For every Dehn twist $D = D(\mathcal{G}, (z_e)_{e \in E(\Gamma)})$, there exists an efficient Dehn twist $D' = D(\mathcal{G}', (z_e)_{e \in E(\Gamma')})$ and an isomorphism between fundamental groups $\rho : \pi_1(\mathcal{G}, w) \rightarrow \pi_1(\mathcal{G}', w')$, where $w \in V(\Gamma)$, $w' \in V(\Gamma')$ are properly chosen vertices, such that $\widehat{D}'\widehat{\rho} = \widehat{\rho}\widehat{D}$.*

(2) *If $D'' = D(\mathcal{G}'', (z_e)_{e \in E(\Gamma'')})$ is a second such efficient Dehn twist, with respect to a analogous fundamental group isomorphism ρ_0 , then there exists*

a graph-of-groups isomorphism $H : \mathcal{G}' \rightarrow \mathcal{G}''$ with $D'' = HD'H^{-1}$ and $\widehat{\rho}_0 = \widehat{H}\widehat{\rho}$.

□

We now return to the issue of H -conjugation as recalled in the last section, but with the specification that the graph-of-groups isomorphism $H : \mathcal{G} \rightarrow \mathcal{G}$ is a Dehn twist D , and we are interested in H -reduced or rather D -reduced elements as discussed in Definition 2.14 and Remark 2.15.

Remark 3.13 (D-reduced v.s. cyclically reduced).

(a) If the Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$ is defined in the classical way, then an element $w \in \Pi(\mathcal{G})$ is D -reduced if and only if it's cyclically reduced. In particular, if for some vertex $v_0 \in V(\mathcal{G})$ an element $w \in \pi_1(\mathcal{G}, v_0) \subset \Pi(\mathcal{G})$ is D -reduced, then w is D -zero if and only if it satisfies $\|w\|_{\mathcal{G}} = 0$.

(b) However, for $w \in \Pi(\mathcal{G})$ which is not D -reduced (or equivalently, not cyclically reduced), its usual conjugates which are cyclically reduced and its D -conjugates which are D -reduced can be very different, in fact they may not even have the same path type.

This shows that even in case of an efficient Dehn twist D for non- D -reduced word in $\Pi(\mathcal{G})$ being D -zero and having its cyclically reduced path length equals to zero are not equivalent.

In view of the existence result for efficient Dehn twist representatives from Theorem 3.12 (1) we will always, when an outer Dehn twist automorphism $\widehat{\varphi} \in \text{Out}(F_n)$ is given without specification of a Dehn twist representative, assume that it is represented by an efficient Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$. Similarly, a Dehn twist automorphism $\varphi \in \text{Out}(F_n)$ without specification of a Dehn twist representative is always assumed to be represented by an efficient Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$.

Recall from Remark 3.5 that the last convention may appear slightly restrictive, in particular when it comes to a partial Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$ relative to a subset \mathcal{V} of vertices v_i of \mathcal{G} for which the induced vertex group automorphism D_{v_i} is known to induce an outer Dehn twist automorphism \widehat{D}_{v_i} . However, it follows from Remark 3.8 that this restriction is immaterial; this class of automorphisms is precisely the same as the one given in Definition 3.7 (b), i.e. the class of partial Dehn twists relative to a family of local Dehn twists.

In view of the uniqueness of D affirmed by part (2) of Theorem 3.12, the following notion is well defined:

Definition 3.14. Let $\varphi \in \text{Aut}(F_n)$ be a Dehn twist automorphism. Then any element $w \in F_n$ is called φ -reduced (or φ -zero) if it is D -reduced (or D -zero) with respect to some efficient Dehn twist representative of φ .

4. CANCELLATION RESULTS

4.1. Growth type.

We first recall some standard notation and well know elementary facts:

Definition 4.1. Let G be a finitely generated group, and let $X = \{x_1, x_2, \dots, x_n\}$ denote its generating set. The *length function* with respect to the generating set X is defined by setting for any $g \in G$:

$$|g|_X := \min\{k \geq 0 \mid g = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_k}^{\varepsilon_k}, i_j \in \{1, \dots, n\}, \varepsilon_j \in \{\pm 1\}\}.$$

The *cyclic length* of $g \in G$ is defined by

$$\|g\|_X := \min\{|hgh^{-1}|_X \mid h \in G\}.$$

Remark 4.2.

- (1) For any $g \in G$ we have $|g|_X \geq 0$, and $|g|_X = 0$ holds if and only if $g = 1$.
- (2) For any $g \in G$ the cyclic length $\|g\|_X$ is the minimum of all lengths of elements in the conjugacy class $[g]$. The elements $h \in G$ and hgh^{-1} such that $|g|_X = |hgh^{-1}|_X$ may not be unique.
- (3) If $G = F_n$ and X is a basis, we also have

$$\|g\|_X = |gg|_X - |g|_X.$$

Furthermore $\|g\|_X = |g|_X$ if and only if $g \in G$ is cyclically reduced.

- (4) For any words $g, h \in F_n$ we always have $|gh| \leq |g| + |h|$.

Remark 4.3. For two sets of generators $X = \{x_1, x_2, \dots, x_n\}$ and $X' = \{x'_1, x'_2, \dots, x'_n\}$ of G , the length functions $|\cdot|_X$ and $|\cdot|_{X'}$ are equivalent up to a constant. To be more precise, there exists a constant $C > 0$ such that for all $g \in G$:

$$\frac{1}{C}|g|_X \leq |g|_{X'} \leq C|g|_X.$$

Definition 4.4. Let $\varphi \in \text{Aut}(G)$ be an automorphism and X be any generating system of G . For any element $g \in G$ we introduce a function $Gr(\varphi, g)$ to trace the length of g under the iteration of φ :

$$\begin{aligned} Gr(\varphi, g)(n) : \mathbb{N} &\rightarrow \mathbb{N} \\ n &\mapsto |\varphi^n(g)|_X \end{aligned}$$

Similarly, for the cyclic length we have:

$$\begin{aligned} Gr_c(\varphi, g)(n) : \mathbb{N} &\rightarrow \mathbb{N} \\ n &\mapsto \|\varphi^n(g)\|_X \end{aligned}$$

Notice that for $\varphi \in \text{Aut}(G)$, $g_1, g_2 \in [g]$, and $n \in \mathbb{N}$ one has:

$$Gr_c(\varphi, g_1)(n) = Gr_c(\varphi, g_2)(n)$$

Also, for $\widehat{\varphi} \in \text{Out}(G)$ and $\varphi_1, \varphi_2 \in \widehat{\varphi}$ we obtain

$$Gr_c(\varphi_1, g)(n) = Gr_c(\varphi_2, g)(n)$$

for all $g \in G$, $n \in \mathbb{N}$. Thus it makes sense to consider the cyclic length of the conjugacy class $[g]$ (or equivalently $g \in [g]$) under the iteration of the outer automorphism $\widehat{\varphi} \in \text{Out}(G)$.

Definition 4.5 (growth type). (a) We say that $g \in G$ *grows at most polynomially of degree d* under iteration of $\varphi \in \text{Aut}(G)$ if $\text{Gr}(\varphi, g)$ is bounded above by a polynomial of degree d . The conjugacy class $[g]$ *grows at most polynomially of degree d* under iteration of $\widehat{\varphi} \in \text{Out}(G)$ (or equivalently, of $\varphi \in \widehat{\varphi}$) if $\text{Gr}_c(\widehat{\varphi}, [g])$ (or $\text{Gr}_c(\varphi, [g])$) is bounded above by a polynomial of degree d .

The automorphism $\varphi \in \text{Aut}(G)$ *has at most polynomial growth of degree d* if any $g \in G$ grows at most polynomially of degree d . Similarly, the outer automorphism $\widehat{\varphi} \in \text{Aut}(G)$ *has at most polynomial growth of degree d* if any $[g] \subset G$ grows at most polynomially of degree d .

(b) Similarly we say that $g \in G$ (or $[g]$) *grows at least polynomially of degree d* under iteration of $\varphi \in \text{Aut}(G)$ (or of $\widehat{\varphi} \in \text{Out}(G)$ respectively) if $\text{Gr}(\varphi, g)$ (or $\text{Gr}_c(\widehat{\varphi}, [g])$) is bounded below by a polynomial of degree d with positive leading coefficient.

The automorphism $\varphi \in \text{Aut}(G)$ *has at least polynomial growth of degree d* if some $g \in G$ grows at least polynomially of degree d . Similarly, the outer automorphism $\widehat{\varphi} \in \text{Aut}(G)$ *has at least polynomial growth of degree d* if some $[g] \subset G$ grows at least polynomially of degree d .

(c) If g (or $[g]$) grows both at most and at least polynomially of degree d , then we say that it grows polynomially of degree d .

(d) An automorphism $\varphi \in \text{Aut}(G)$ (or an outer automorphism $\widehat{\varphi} \in \text{Out}(G)$) *grows polynomially of degree d* if every $g \in G$ (or every $[g] \subset G$) grows at most polynomially of degree d and in particular there exists an element $g_0 \in G$ (or a conjugacy class $[g] \subset G$) that grows polynomially of degree d .

Remark 4.6. Because the length functions with respect to different generating systems are equivalent up to a constant, the definitions about growth types given in Definition 4.5 are independent of the chosen generating system of G .

Definition 4.7. Let $(w_k)_{k=1}^{+\infty}$ a family of elements in G , we sometimes say that the sequence w_k *grows at least polynomially of degree d* if there exists constant $C_1 > 0$ such that $C_1 k^d \leq |w_k|$. Similarly, we say that w_k *grows at most polynomially of degree d* if there exists $C_2 > 0$ such that $|w_k| \leq C_2 k^d$ and that w_k *grows polynomially of degree d* if one can find $C_1, C_2 > 0$ such that $C_1 k^d \leq |w_k| \leq C_2 k^d$.

4.2. Cancellation and iterated products.

Let F_n be a free group and denote by \mathcal{A} a fixed basis of F_n . As before, we denote the combinatorial length (with respect to \mathcal{A}) of an element W by $|W| = |W|_{\mathcal{A}}$, and the cyclically reduced length of W by $\|W\| = \|W\|_{\mathcal{A}}$.

Lemma 4.8. *Let F_n be a free group, let $V \subset F_n$ be a subgroup of rank $n \geq 2$, and let $(w_i)_{i \in \mathbb{N}}$ be any infinite family of elements $w_i \in F_n$. Then for any basis \mathcal{A} of F_n there exists an element $v \in V$ and a constant $C \geq 0$ such that for infinitely many indices $i \in \mathbb{N}$ the cancellation in the product*

$|w_i v w_i^{-1}|_{\mathcal{A}}$ is bounded, i.e. one has:

$$|w_i v w_i^{-1}|_{\mathcal{A}} \geq |w_i|_{\mathcal{A}} + |v|_{\mathcal{A}} + |w_i^{-1}|_{\mathcal{A}} - C = 2|w_i|_{\mathcal{A}} + |v|_{\mathcal{A}} - C$$

Proof. Pick elements v_1 and v_2 in V which generate a subgroup of rank 2. Consider the products $w_i v_1^m w_i^{-1}$, for increasing integers m . We observe that one of the following must hold:

- (1) For some $m \in \mathbb{N}$ the cancellation in $w_i v_1^m w_i^{-1}$ is bounded uniformly with respect to all $i \in \mathbb{N}$.
- (2) For any $m \in \mathbb{N}$ there is an index $j(m) \in \mathbb{N}$ such that $w_{j(m)}$ has the suffix v_1^{-m} .
- (3) For any $m \in \mathbb{N}$ there is an index $j'(m) \in \mathbb{N}$ such that $w_{j'(m)}^{-1}$ has the prefix v_1^{-m} , or equivalently, $w_{j'(m)}$ has the suffix v_1^m .

In the case of alternative (1), the proof is completed. In case of (2), we replace the family of all w_i by the subfamily of all w_j with $j = j(m)$ for any $m \in \mathbb{N}$. It follows from the statement (2) that this subfamily is infinite. In case (3) we do the same, but with $j = j'(m)$.

We now repeat the above trichotomy, with w_i replaced by w_j , and with v_1 replaced by v_2 . We observe that in this second trichotomy the alternatives (2) and (3) lead to elements w_j with suffix that is simultaneously an arbitrary large positive or negative power of v_1 and of v_2 . But this is impossible, by our assumption that v_1 and v_2 generate a subgroup of rank 2. Thus alternative (1) must hold for the second trichotomy, which proves our claim. \square

4.3. Cancellation in long products.

Definition 4.9. We say U and V admit a common root if there exist an element $R \in F_n$, $R \neq 1$, such that $U = R^{m_1}$, $V = R^{m_2}$ for some suitable $m_1, m_2 \in \mathbb{N}$, R is called a *common root* of U and V .

We recall the following well known fact:

Proposition 4.10 ([5]). *For any elements $U, V \in F_n$, there is an algorithm which decides whether they admit a common root.*

The following is well-known too:

Lemma 4.11. *For two elements $U, V \in F_n$ we have $U^{n_1} \neq V^{n_2}$ for any $n_1, n_2 \geq 1$ if and only if U, V don't admit a common root.*

Proof. On one hand, if U, V admit a common root R such that $U = R^{m_1}$, $V = R^{m_2}$ for some $m_1, m_2 \geq 1$, then we have $U^{m_2} = V^{m_1}$.

On the other hand, if there exist $n_1, n_2 \geq 1$ such that $U^{n_1} = V^{n_2}$, by comparing the suffixes and prefixes of U and V , we can find a common root $R \in F_n$. \square

Lemma 4.12. *If $U^{-1}, V \in F_n$ don't admit a common root, then the cancellation of the products $U^{n_1} V^{n_2}$, for any $n_1, n_2 \in \mathbb{N}$, is uniformly bounded.*

As consequence, there exists a constant and $K_0 = K(U^{-1}, V)$ such that for any $n_1, n_2 \in \mathbb{N}$, we have:

$$|U^{n_1}V^{n_2}| \geq n_1\|U\| + n_2\|V\| + K_0$$

Proof. If no constant K_0 as postulated exists, then by comparing the subfixes and prefixes of U and V we can find a common root $R \in F_n$ for U^{-1} and V , which contradicts our hypothesis.

Therefore by definition we can find $B_1 \geq 0$ such that

$$|U^{n_1}| + |V^{n_2}| - |U^{n_1}V^{n_2}| \leq B_1.$$

Hence, by taking $K_0 = -B_0$ we have

$$|U^{n_1}V^{n_2}| \geq |U^{n_1}| + |V^{n_2}| + K_0 \geq n_1\|U\| + n_2\|V\| + K_0$$

for any $n_1, n_2 \geq 0$. □

Remark 4.13. Furthermore, for $U^{-1}, V \in F_n$ which don't admit a common root, we have at the same time that the cancellation of the products $V^{n_2}U^{n_1}$, for any $n_1, n_2 \in \mathbb{N}$, is uniformly bounded by some constant $B_2 \geq 0$. Taking $K = -B_1 - B_2$, we have immediately

$$\|U^{n_1}V^{n_2}\| \geq n_1\|U\| + n_2\|V\| + K.$$

Lemma 4.14. Let $X, b, Y \in F_n$ be elements such that $X^{-m_1} \neq bY^{m_2}b^{-1}$, for any $m_1, m_2 \geq 1$. Then there exists a constant $K = K(X, b, Y)$ such that for any $n_1, n_2 \geq 0$ we have:

$$|X^{n_1}bY^{n_2}| \geq n_1\|X\| + n_2\|Y\| + K.$$

Proof. For any $n_1, n_2 \geq 0$, we may consider the word

$$X^{n_1}bY^{n_2} = X^{n_1}bY^{n_2}b^{-1}b = X^{n_1}(bYb^{-1})^{n_2}b.$$

Taking $U = X$, $V = bYb^{-1}$, we know from Lemma 4.11 that U^{-1}, V don't admit a common root. Hence it follows from Lemma 4.12 that there exists $K_0 = K(U^{-1}, V)$ such that

$$|U^{n_1}V^{n_2}| \geq n_1\|U\| + n_2\|V\| + K_0.$$

Let $K = K_0 - |b|$, we have immediately the inequality

$$\begin{aligned} |X^{n_1}bY^{n_2}| &= |U^{n_1}V^{n_2}b| \\ &\geq |U^{n_1}V^{n_2}| - |b| \geq n_1\|U\| + n_2\|V\| + K = n_1\|X\| + n_2\|Y\| + K. \end{aligned}$$

□

Remark 4.15. Please notice that in the situation considered in Lemma 4.14 we may *not* have the following inequality:

$$\|X^{n_1}bY^{n_2}\| \geq n_1\|X\| + n_2\|Y\| + K$$

Counter-example: Consider $F_2 = \langle a, b \rangle$ and let $X = a^{-1}$, $Y = a$.

Lemma 4.16. *Let $X, b, Y \in F_n$ be elements such that $X^{-m_1} \neq bY^{m_2}b^{-1}$, for any $m_1, m_2 \geq 1$. Then there exist cyclically reduced conjugates \tilde{X}, \tilde{Y} of X, Y and $n_0 \in \mathbb{N}$ such that for all $n_1, n_2 \geq n_0$, in the reduced product of $X^{n_1}bY^{n_2}$ neither \tilde{X} nor \tilde{Y} is completely cancelled.*

Proof. Consider the cyclically reduced conjugates $\tilde{X} = w_1Xw_1^{-1}, \tilde{Y} = w_2Yw_2^{-1}$ for X, Y , where $w_1, w_2 \in F_n$. We may then write the word

$$X^{n_1}bY^{n_2} = w_1^{-1}\tilde{X}^{n_1}w_1bw_2^{-1}\tilde{Y}^{n_2}w_2^{-1}.$$

We derive from the uniformly bounded property that there exists $n_0 \in \mathbb{N}$ such that when $n_1, n_2 \geq n_0$, in the reduced product of $X^{n_1}bY^{n_2} = w_1^{-1}\tilde{X}^{n_1}w_1bw_2^{-1}\tilde{Y}^{n_2}w_2^{-1}$, neither \tilde{X} nor \tilde{Y} is completely cancelled.

More concretely, since we always have the inequality

$$|X^{n_1}bY^{n_2}| \leq n_1\|X\| + n_2\|Y\| + 2|w_1| + 2|w_2| + |b|,$$

it follows from Lemma 4.14 that the cancellation in the products $X^{n_1}bY^{n_2}$ is uniformly bounded by $B = 2|w_1| + 2|w_2| + |b| - K$, where $K = K(X, b, Y)$ is the constant obtained in Lemma 4.14. Then we may choose $n_0 \in \mathbb{N}$ that satisfies $\|X^{n_0}\| = n_0\|X\|, \|Y^{n_0}\| = n_0\|Y\| \geq B$. □

4.4. Main cancellation result.

Let now \mathcal{F} be a set which consists of finitely many triplets (X_i, b_j, X_k) , where $X_i, b_j, X_k \in F_n$ are elements which satisfy

- (1) $\|X_i\| > 0$ and $\|X_k\| > 0$, and
- (2) $X_i^{-m_1} \neq b_jX_k^{m_2}b_j^{-1}$, for any $m_1 \geq 1, m_2 \geq 1$.

We consider below words

$$w = w(n_1, n_2, \dots, n_q) = c_0y_1^{n_1}c_1y_2^{n_2}c_2 \dots c_{q-1}y_q^{n_q}c_q \in F_n$$

which have the property $(y_i, c_i, y_{i+1}) \in \mathcal{F}$, for $1 \leq i \leq q$.

We then derive the following proposition:

Proposition 4.17. *There exist constants $N_0 \geq 0$ and K_0 such that for any $w = w(n_1, n_2, \dots, n_q) \in F_n$ as above and any $n_i \geq N_0$ (with $1 \leq i \leq q$), we have*

$$|w| \geq \sum_{i=1}^q n_i\|y_i\| + (q-1)K_0.$$

Proof. It follows directly from Lemma 4.14 and Remark 4.16 that for each triplet (y_i, c_i, y_{i+1}) , $1 \leq i \leq q-1$, there exist constants $K_i = K(y_i, c_i, y_{i+1})$ and $N_i \geq 0$ such that :

$$|y_i^{n_i}c_iy_{i+1}^{n_{i+1}}| \geq n_i\|y_i\| + n_{i+1}\|y_{i+1}\| + K_i,$$

Moreover, if $n_i, n_{i+1} \geq N_i$ neither of the cyclically reduced conjugates $\tilde{y}_i = w_iy_iw_i^{-1}$, $\tilde{y}_i + 1 = w_{i+1}y_{i+1}w_{i+1}^{-1}$ is completely cancelled in the reduced product

$$y_i^{n_i}c_iy_{i+1}^{n_{i+1}} = w_i^{-1}y_i^{n_i}w_ic_iw_{i+1}^{-1}y_{i+1}^{n_{i+1}}w_{i+1}.$$

We now prove the proposition by induction.

- (1) The case for $q = 1$ is trivial while the case for $q = 2$ is shown in Lemma 4.14.
- (2) Suppose the inequality holds for $q = s$. In other words: we can find constants $N \geq N_{s-1}$ and K such that for $n_i \geq N$

$$|c_0 y_1^{n_1} c_1 y_2^{n_2} c_2 \dots c_{s-1} y_s^{n_s} c_s| \geq \sum_{i=1}^s n_i \|y_i\| + (s-1)K$$

and $\tilde{y}_s^{n_s}$ is not completely cancelled in the reduced procedure.

In particular, given that in each inductive step the constants N and K' depends only on the triplets (y_i, c_i, y_{i+1}) , for $1 \leq i \leq q-1$ one can in fact deduce the final cancellation bound $(q-1)K_0$ based on just the family \mathcal{F} . In other words, the cancellation bound K_0 doesn't depend on the exponents n_i 's, once they are bigger than $N_0 := N$.

□

Remark 4.18. In addition if $(y_q, c_q c_0, y_1) \in \mathcal{F}$, similarly to what is done in the last proof, we may apply the same technique to the triplet $(y_q, c_q c_0, y_1)$ and obtain the following estimate for cyclical length of w (again assuming $n_i \geq N_0$ for all exponents n_i):

$$\|w\| \geq \sum_{i=1}^q n_i \|y_i\| + qK_0$$

4.5. Cancellation bounds for \mathcal{T} -products.

Let $\mathcal{T} \subset F_n \setminus \{1\}$ be a finite set. We say that a product

$$(4.1) \quad W = W(w_i, y_i) = w_0 y_1 w_1 y_2 w_2 \dots w_{q-1} y_q w_q$$

is a \mathcal{T} -word of \mathcal{T} -length

$$|W|_{\mathcal{T}} = q,$$

if $w_i \in F_n$ and $y_i \in \mathcal{T}$ for all indices i , and we say that the product W is \mathcal{T} -reduced if $y_i^{-m} \neq w_i y_{i+1}^{m'} w_i^{-1}$ for any integers $m, m' \geq 0$.

For any \mathcal{T} -word W as in (4.1) and any multi-exponent

$$(4.2) \quad [n] = (n_1, n_2, \dots, n_q) \in \mathbb{N}^q$$

we denote by $W^{[n]}$ the word

$$W^{[n]} = w_0 y_1^{n_1} w_1 y_2^{n_2} w_2 \dots w_{q-1} y_q^{n_q} w_q.$$

For any $n_0 \in \mathbb{Z}$ we write

$$[n] \geq n_0$$

if $n_i \geq n_0$ holds for all components n_i of $[n]$.

Proposition 4.19. (a) For any reduced \mathcal{T} -word W as in (4.1) and any basis \mathcal{A} of F_n there exist constants $K = K(W) \geq 0$ and $n_0 \in \mathbb{Z}$ such that for any multi-exponent $[n] \in \mathbb{N}^q$ as in (4.2), with $[n] \geq n_0$, the \mathcal{A} -lengths satisfy

$$|W^{[n]}|_{\mathcal{A}} \geq \sum_{k=1}^q n_k \|y_k\|_{\mathcal{A}} - K$$

(b) If all of the intermediate words w_i in W belong to a finite family \mathcal{W} , then the above constant K can be taken to be equal to $(q-1)K_0$ for some constant $K_0 \geq 0$ which only depends on \mathcal{W} and \mathcal{T} -length, but not on the multi-exponent $[n]$. Similarly, the constant n_0 can be chosen to depend only on \mathcal{W} .

Proof. For the given family $W = W(w_i, y_i)$ we set

$$\mathcal{F}(W) = \{(y, w, y') \mid w \in \mathcal{W}, y, y' \in \mathcal{T}, y^{-m} \neq wy^m w^{-1} \text{ if } m, n \geq 0\}$$

Then our claim follows directly from Proposition 4.17. \square

5. GRAPH-OF-GROUPS WITH TRIVIAL EDGE GROUPS: GROWTH BOUNDS

In this section we will suppress the base point v in the fundamental group $\pi_1 \mathcal{G}$ of a graph-of-groups \mathcal{G} if it is immaterial, and write simply $\pi_1 \mathcal{G}$.

Lemma 5.1. Let \mathcal{G} be a graph-of-groups with trivial edge groups. Let $(W_i)_{i=1}^{+\infty} \subset \pi_1(\mathcal{G})$ be a family of cyclically reduced words on \mathcal{G} , where $W_i = v_0(i)t_1v_1(i)t_2\dots t_qv_q(i)$. If for some $1 \leq k \leq q$, the length of $v_k(i)$ under some (hence any) finite generating system \mathcal{B}_k of the vertex group G_{v_k} , i.e. $|v_k(i)|_{\mathcal{B}_k}$, grows quadratically with respect to i , then $\|W_i\|$ grows at least quadratically with respect to i .

Proof. As shown in Section 2.1 that the union of generating systems of each vertex group $\bigcup_{v \in V(\mathcal{G})} \mathcal{B}_v$ forms a subset of a generating system \mathcal{B} of $\pi_1(\mathcal{G})$,

where each vertex group is a free factor.

We obtain:

$$\|W_i\|_{\mathcal{B}} \geq \sum_{k=1}^{q-1} |v_k(i)|_{\mathcal{B}_k} + |v_q(i)v_0(i)|_{\mathcal{B}_0}$$

It follows immediately from the conditions that at least one of $|v_k(i)|_{\mathcal{B}_k}$ grows quadratically and that W_i is cyclically reduced that the cyclically reduced length of W_i grows at least quadratically. \square

Recall that a graph-of-groups \mathcal{G} is called *minimal* if it doesn't contain a proper subgraph \mathcal{G}' such that the inclusion induces an isomorphism on the fundamental groups. For a finite graph-of-groups with trivial edge groups this amounts to requiring that any vertex of valence 1 has a non-trivial vertex group.

Lemma 5.2. *Let \mathcal{G} be a minimal graph-of-groups with trivial edge groups, Then for any edge e with terminal vertex $v = \tau(e)$ that has non-trivial vertex group G_v one can find a cyclically reduced word $w \in \pi_1(\mathcal{G})$ with underlying path that runs subsequently through the edge e and directly after through \bar{e} .*

Proof. Because \mathcal{G} is a minimal graph-of-groups, each connected component of the graph Γ' obtained from $\Gamma(\mathcal{G})$ by removing the edge e must contain either a circuit ω or else a vertex v' with non-trivial vertex group $G_{v'}$. Let γ be a path in Γ' which connects $\iota(e)$ either to the initial (= terminal) vertex of some such ω , or else to v' . Let $v \in G_v \setminus \{1\}$ and $u \in G_{v'} \setminus \{1\}$ (in the second case only). Then $\gamma_*^{-1}ev\bar{e}\gamma_*\omega_*$ (in the first case) or $\gamma_*^{-1}ev\bar{e}\gamma_*u$ (in the second one) are the words we are looking for, where γ_* and ω_* denote the sequence of stable letters t_{e_i} defined by the edges e_i of γ and ω respectively. \square

Proposition 5.3. *Let \mathcal{G} be a minimal graph-of-groups with trivial edge groups, and let $H : \mathcal{G} \rightarrow \mathcal{G}$ be a graph-of-groups automorphism which acts trivially on the underlying graph $\Gamma = \Gamma(\mathcal{G})$.*

Let v be a vertex of Γ , with vertex group automorphism $H_v : G_v \rightarrow G_v$. For some edge e with terminal vertex $\tau(e) = v$ denote by $\delta_e \in G_v$ the correction term of e .

Assume that for some $g \in G_v$ there exist a constant $C > 0$ and a strictly increasing sequence of numbers $n_i \in \mathbb{Z}$ which satisfy:

$$|H_v^{(n_i)}(\delta_e^{-1})H_v^{n_i}(g)H_v^{(n_i)}(\delta_e^{-1})^{-1}| \geq Cn_i^2$$

Then the induced outer automorphism \hat{H} of $\pi_1\mathcal{G}$ has at least quadratic growth.

Proof. It follows immediately from Lemma 5.2 that one can find cyclically reduced word $w \in \pi_1\mathcal{G}$ with underlying path that runs through the edge e and subsequently through \bar{e} , and w contains the word $t_e g t_e^{-1}$ as subword.

As a consequence the iteration of $H_* : \Pi(\mathcal{G}) \rightarrow \Pi(\mathcal{G})$ on w will give words $H_*^k(w)$ that contain

$$t_e H_v^{(k)}(\delta_e^{-1}) H_v^k(g) H_v^{(k)}(\delta_e^{-1})^{-1} t_e^{-1}$$

as subword.

Hence it follows from the assumed inequality and from Lemma 5.1 that the subsequence $\|H_*^{n_i}(w)\|$ grows at least quadratically. Therefore the conjugacy class of w grows at least quadratically under the iteration of H_* , which implies that the induced outer automorphism \hat{H} grow at least quadratically. \square

6. DEHN TWISTS

This section is dedicated to translate our cancellation propositions into graph-of-groups language.

Through the whole section, we always assume that the free group F_n is of rank $n \geq 2$.

We first prove the following Proposition.

Proposition 6.1. *Let F_n be a free group with rank $n \geq 2$, and let $\mathcal{D} \in \text{Aut}(F_n)$ be a Dehn twist automorphism which is represented by an efficient Dehn twist. Then we have:*

(1) *There exists a finite set of “twistors” $\mathcal{T} = \{z_1, \dots, z_r\} \subset F_n \setminus \{1\}$, such that for any element $w \in F_n$ there exists a (non-unique) “ \mathcal{T} -decomposition” of w as product*

$$(6.1) \quad w = w_0 w_1 w_2 \dots w_{q-1} w_q = w_0 y_1^0 w_1 y_2^0 w_2 \dots w_{q-1} y_q^0 w_q$$

with $w_i \in F_n$ and y_i or y_i^{-1} in \mathcal{T} such that

$$\mathcal{D}^n(w) = w_0 y_1^n w_1 y_2^n w_2 \dots w_{q-1} y_q^n w_q,$$

and $y_i^{-m} \neq w_i y_{i+1}^{m'} w_i^{-1}$ for any integers $m, m' \geq 0$.

(2) *The rank of the subgroup of F_n which consists of all elements fixed by \mathcal{D} satisfies:*

$$\text{rk}(\text{Fix}(\mathcal{D})) \geq 2$$

Proof. By definition \mathcal{D} is represented by an efficient Dehn twist $D : \mathcal{G} \rightarrow \mathcal{G}$ on a graph-of-groups \mathcal{G} with fundamental group $\pi_1 \mathcal{G}$ isomorphic to F_n . We pick a vertex v_0 of \mathcal{G} as base point and specify the above isomorphism to be $\theta : F_n \xrightarrow{\cong} \pi_1(\mathcal{G}, v_0)$. The automorphism \mathcal{D} fixes the θ^{-1} -image of the vertex group G_{v_0} of \mathcal{G} elementwise, and since efficient Dehn twists have all vertex groups of rank ≥ 2 (see Proposition 3.10), this proves claim (2) of the proposition.

In order to obtain claim (1), we chose a maximal tree Y in the graph Γ and identify in the usual fashion each vertex group G_v canonically with a subgroup of $\pi_1(\mathcal{G}, v_0)$ by connecting v_0 to v through a simple path in Y . Similarly, for any edge e the stable letter $t_e \in \Pi(\mathcal{G})$ gives rise to an element in $\pi_1(\mathcal{G}, v_0)$ by connecting v_0 to the terminal vertices of e through simple paths in Y (which gives $1 \in \pi_1(\mathcal{G}, v_0)$ if and only if e belongs to Y).

The collection \mathcal{T} is then given by the twistors z_e of the edges $e \in E^+(\mathcal{G})$ (for some orientation $E^+(\mathcal{G}) \subset E(\mathcal{G})$, see subsection 2.1). For any $w \in F_n$ the collection of factors w_i in the \mathcal{T} -product decomposition (6.1) is obtained by writing $\theta(w)$ as a reduced word $v_0 t_1 v_1 t_2 v_3 \dots v_{q-1} t_q v_q$ in $\pi_1(\mathcal{G}, v_0)$ (see Proposition 2.6), and by applying θ^{-1} to v_0 or to any of the $t_i v_i$ for $i = 1, \dots, q$. The equality $\mathcal{D}^n(w) = w_0 y_1^n w_1 y_2^n w_2 \dots w_{q-1} y_q^n w_q$ is an immediate consequence of the definition of a Dehn twist automorphism, see Remark 3.3. The inequalities $y_i^{-m} \neq w_i y_{i+1}^{m'} w_i^{-1}$, for any integers $m, m' \geq 0$, follow directly from the condition that the efficient Dehn twist D does not have twistors that are positively bonded, see Definition 3.9 (5). \square

Remark 6.2. The representation of $w \in F_n$ as \mathcal{T} -product as given in Proposition 6.1 is not unique. However, it follows from the proof that the sequence of twistors y_i is well defined, and thus also the \mathcal{T} -length $|w|_{\mathcal{T}}$ of w . The intermediate words w_i are well defined up to replacing them by $y_i^p w_i y_{i+1}^q$ for some $p, q \in \mathbb{Z}$.

Definition 6.3. A \mathcal{T} -product representative of $w \in F_n$ as in Proposition 6.1 is called *cyclically \mathcal{T} -reduced* if $y_q^{-m} \neq w_q w_0 y_1^{m'} w_0^{-1} w_q^{-1}$ for any integers $m, m' \geq 0$.

Definition-Remark 6.4.

(1) Let \mathcal{D} and \mathcal{T} be as in Proposition 6.1. Assume that $D : \mathcal{G} \rightarrow \mathcal{G}$ is an efficient Dehn twist representative of \mathcal{D} , with respect to some identification isomorphism $\theta : F_n \rightarrow \pi_1(\mathcal{G}, v_0)$ for some vertex v_0 of \mathcal{G} .

For some element $w \in F_n$ let $W \in \Pi(\mathcal{G})$ be the corresponding element in the Bass group of \mathcal{G} , i.e. $W = \theta(w) \in \pi_1(\mathcal{G}, v_0) \subset \Pi(\mathcal{G})$.

We say that w is *\mathcal{D} -reduced* if W is D -reduced.

(2) Recall that $\theta(w)$ is called D -reduced if the element W is reduced as word in $\Pi(\mathcal{G})$, and if its \mathcal{G} -length can not be shortened by D -conjugation, i.e. by passing over to a word $V^{-1}WD(V) \in \pi_1(\mathcal{G}, v_1) \subset \Pi(\mathcal{G})$, for some vertex v_1 of \mathcal{G} . It follows from our considerations in the proof of Proposition 6.1 that in this case w is \mathcal{T} -reduced and cyclically \mathcal{T} -reduced.

(3) If W is not D -reduced, then we can follow the procedure indicated in Definition 2.14 and Remark 2.15 to perform iteratively elementary D -reductions until we obtain a new word $W' \in \pi_1(\mathcal{G}, v_1) \subset \Pi(\mathcal{G})$ which is D -reduced, for a possibly different vertex v_1 . In this case we change our identification isomorphism θ corresponding to the performed D -reductions to obtain a new identification $\theta' : F_n \rightarrow \pi_1(\mathcal{G}, v_1)$ with respect to which w is D -reduced.

(4) Recall that a D -reduced word $W \in \Pi(\mathcal{G})$ is D -zero if and only if W has \mathcal{G} -length 0, or in other words, W is contained in some vertex group G_v . In this case we say that a word $w \in F_n$ with $\theta(w) = W$ is *\mathcal{D} -zero*.

We thus note that any $w \in F_n$ which is \mathcal{D} -reduced (possibly with respect to a modified identification isomorphism θ' as in (3) above) but not \mathcal{D} -zero is \mathcal{T} -reduced, cyclically \mathcal{T} -reduced, and of \mathcal{T} -length $|w|_{\mathcal{T}} \geq 1$.

Let $\mathcal{D} \in \text{Aut}(F_n)$ be a Dehn twist automorphism, recall that for any element $w \in F_n$ and any integer $n \geq 1$ we denote by $\mathcal{D}^{(n)}(w)$ the *iterated product*, defined through:

$$\mathcal{D}^{(k)}(w) := w \mathcal{D}(w) \mathcal{D}^2(w) \dots \mathcal{D}^{k-1}(w),$$

and the *partial iterated product* is given by

$$\mathcal{D}^{(k_1, k_2)}(w) := \mathcal{D}^{k_1}(w) \mathcal{D}^{k_1+1}(w) \dots \mathcal{D}^{k_2}(w).$$

Proposition 6.5. Let $\mathcal{D} \in \text{Aut}(F_n)$ be a Dehn twist automorphism, and denote by \mathcal{T} the set of “twistors” defined in Proposition 6.1. Let $w \in F_n$ be a D -reduced word which is not \mathcal{D} -zero, i.e. $|w|_{\mathcal{T}} \neq 0$. Then the combinatorial length of $\mathcal{D}^{(k)}(w)$ has quadratic growth with respect to k , i.e. one can find constant C_1 such that

$$|\mathcal{D}^{(k)}(w)| \simeq C_1 k^2.$$

Proof. It follows from Proposition 6.1 that w admits a \mathcal{T} -reduced decomposition $w = w_0 y_1^0 w_1 y_2^0 w_2 \dots w_{q-1} y_q^0 w_q$ with $w_i \in F_n$ and $y_i \in \mathcal{T}$ which satisfies that $\mathcal{D}^k(w) = c_0 y_1^k c_1 \dots c_{q-1} y_q^k c_q$.

It follows immediately from Proposition 4.19 that there exist constants $N_0 \geq 0$ and K_0 such that for $k \geq N_0$ we have:

$$|\mathcal{D}^k(w)| \geq \sum_{i=1}^q k \|y_i\| + (q-1)K_0.$$

Since w is \mathcal{D} -reduced, Remark 6.4 shows that the cancellation in between $\mathcal{D}^i(w)\mathcal{D}^{i+1}(w)$ is bounded by some constant K_1 .

By taking $K'_0 = \min\{K_0, K_1\}$, we obtain that, for $N_0 \leq k' \leq k$,

$$|\mathcal{D}^{(k',k)}(w)| \geq \sum_{j=k'}^k \sum_{i=1}^q j \|y_i\| + (kq - k'q - 1)K'_0.$$

Since $q \geq 1$, the cancellation bounds $(kq - k'q - 1)K'_0$ grows linearly with respect to k .

In particular, we may take $k' = N_0$ hence

$$\begin{aligned} |\mathcal{D}^{(N_0,k)}(w)| &\geq \sum_{j=N_0}^k \sum_{i=1}^q j \|y_i\| + (kq - N_0q - 1)K'_0 \\ &= \frac{k(k - N_0)}{2} \sum_{i=1}^q \|y_i\| + (kq - N_0q - 1)K'_0 \end{aligned}$$

and

$$|\mathcal{D}^{(k)}(w)| \geq \frac{k(k-1)}{2} \sum_{i=1}^q \|y_i\| + (kq - N_0q - 1)K'_0 - |\mathcal{D}^{(N_0)}(w)|.$$

On the other hand, for all $k \in \mathbb{N}$ we always have

$$|\mathcal{D}^{(k)}(w)| \leq \sum_{j=1}^k \sum_{i=1}^q j |y_i| + \sum_{j=1}^k \sum_{i=0}^q |c_i| = \frac{k(k-1)}{2} \sum_{i=1}^q |y_i| + k \sum_{i=0}^q |c_i|$$

Together these two inequalities give $|\mathcal{D}^{(k)}(w)| \simeq C_1 k^2$ for some $C_1 > 0$. \square

Remark 6.6. Let N_0 be as above. Denote

$$B = \mathcal{D}^{(N_0)}(w) = wD(w)D^2(w) \dots D^{N_0-1}(w).$$

We can now find $N_1 > N_0 \in \mathbb{N}$ large enough so that the following two conditions hold:

1. Let w_1 be the prefix of $\mathcal{D}^{(N_0, N_1)}(w)$ which ends with y_{q-1} such that $|B| \leq |w_1|$.
2. The number N_1 is large enough so that $|y_q^{N_1-1}| \geq |w_1| + |c_q| + |c_{q-1}|$.

These two conditions do guarantee that the cyclic cancellation of $\mathcal{D}^{(N_1)}(w)$ cannot proceed into the subword

$$w_1^{-1}\mathcal{D}^{(N_0, N_1)}(w) = c_{q-1}y_q^{t_0}c_q \dots c_{q-1}y_q^{N_1}c_q$$

further than $c_{q-1}y_q^{t_0}$ on the left and $y_q^{N_1}c_q$ on the right, as otherwise a subword of length $|y_q|$ of $y_q^{t_0}$ would cancel against a subword of $y_q^{N_1}$, which is impossible since $w^{-1} \neq v w v^{-1}$ in F_n , for any $w \neq 1$.

Hence we obtain from the above arguments that also the cyclic length $\|\mathcal{D}^{(k)}(w)\|$ has quadratic growth with respect to k .

7. MAIN RESULT AND THEOREM

The following is a slightly stronger version of what has been stated as Theorem 1.1 in the Introduction.

Theorem 7.1. *Let \mathcal{G} be a minimal graph-of-groups with trivial edge groups, and let $H : \mathcal{G} \rightarrow \mathcal{G}$ be a graph-of-groups automorphism which acts trivially on the underlying graph $\Gamma = \Gamma(\mathcal{G})$.*

Assume that for some vertex v of Γ the vertex group G_v is a free group, that the induced vertex group automorphism $H_v : G_v \rightarrow G_v$ is a Dehn twist automorphism, and that for some edges e of Γ with endpoint $\tau(e) = v$ the H -correction term δ_e is not H_v -zero.

Then the induced outer automorphism \hat{H} of $\pi_1\mathcal{G}$ has at least quadratic growth.

Proof. By Proposition 5.3 it suffices to show that for $w = \delta_e$ one can find an element $u \in G_v$ such that there is a subsequence of $|H_v^{(k)}(w^{-1})H_v^k(u)H_v^{(-k)}(w)|$ which grows quadratically.

Since by assumption the correction term $w = \delta_e$ is not H_v -zero, we can apply Proposition 6.5 to obtain that the length of the iterated product $H_v^{(-k)}(w)$ grows at least quadratically.

One then uses Proposition 6.1 (2) to show that the subgroup $\text{Fix}(H_v)$ of all fixed elements of H_v has rank at least 2. This enables us to find via Lemma 4.8 an element $u \in \text{Fix}(H_v)$ such that for infinitely many indices $n_i \geq 0$ the cancellation in the product $H_v^{(n_i)}(w^{-1})H_v^{n_i}(u)H_v^{(-n_i)}(w) = H_v^{(n_i)}(w^{-1})uH_v^{(-n_i)}(w)$ is uniformly bounded, so that from the previous paragraph we obtain that their lengths grow at least quadratically. \square

Recall from Definition 3.7 (b) that a partial Dehn twist $H : \mathcal{G} \rightarrow \mathcal{G}$ relative to a family of local Dehn twists is given, for any vertex v of \mathcal{G} , through an identification isomorphism $G_v \cong \pi_1\mathcal{G}_v$ and a “local” Dehn twist representatives $D_v : \mathcal{G}_v \rightarrow \mathcal{G}_v$ which we can assume without loss of generality to be efficient. We say that for any edge e of \mathcal{G} the correction term δ_e is *locally zero* if it is $D_{\tau(e)}$ -zero.

Then the last theorem gives directly:

Corollary 7.2. *Let $\hat{\varphi} \in \text{Out}(F_n)$ be represented by a partial Dehn twist $H : \mathcal{G} \rightarrow \mathcal{G}$ relative to a family of local Dehn twists. Assume that for some edge e of \mathcal{G} the correction term δ_e is not locally zero.*

Then $\hat{\varphi}$ has at least quadratic growth. \square

As a final remark we want to point out that Corollary 1.2 is indeed a direct consequence of the above corollary together with the main result of [10], which states that, in the situation of Corollary 7.2, if all of the correction terms for the edges of \mathcal{G} are locally zero, then H can be blown-up at the local Dehn twists to give a Dehn twist representative of the automorphism $\hat{\varphi}$.

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